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AN INVESTIGATION OF THE STABILITY OF THE PERIODIC MOTION OF A RIGID BODY WHEN THERE ARE COLLISIONS WITH A HORIZONTAL PLANE†

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The problem of the orbital stability of the translation-rotational motion of a rigid body in the shape of a circular disk above a fixed horizontal plane in a uniform gravitational field is solved. It is assumed that the plane is absolutely smooth, the disk is thin and homogeneous and the collision of the disk with the plane is absolutely elastic. In the non-perturbed motion the plane of the disk is vertical, the disk rotates with a constant angular velocity about the vertical or horizontal axis and its centre of gravity executes periodic oscillations along the fixed vertical as the result of the collisions.

In the first investigations [1, 2] of the dynamics of a heavy rigid body having an arbitrary central ellipsoid of inertia when there are collisions with an absolutely smooth plane, the equations of a free motion of the body in the finite time intervals between the collisions and fitting of the boundary conditions [3] at the ends of these intervals were used. The boundary conditions were derived from the general theory of frictionless collision [4]. Assuming that the plane is stationary and the collision is absolutely elastic, the stability in the first approximation of the rotation of the body about a principal central axis of inertia was investigated in [1], and the peculiar fact that the domains of stability and instability in the height of the jump of the body under the plane were "quantized" was found. These results were carried over to the case in which the collision is not elastic and the plane executes specified sinusoidal vibrations along the vertical direction [2].

To investigate the motions of systems with an ideal non-restoring constraint the non-smooth change of variables which eliminates the non-restoring constraint and makes it possible to derive the equations of motion in the form of Routh's equations, which hold in an arbitrary time interval, was proposed in [5]. This made it possible to solve a set of problems in the dynamics of vibrational-collision systems using the averaging method [6, 7].

The combination of the non-smooth change of variables from [5] and the special choice of generalized coordinates, which is realized using a "reducing" change of variables made it possible [8] to write the equations of motion of a system with an ideal non-restoring constraint in the form of Hamilton's equations. This approach was then used when investigating problems of the stability of the motion of a body when it collides with a horizontal plane and in a qualitative analysis of its dynamics by means of the Poincaré and the KAM theory methods [9–13].

In this paper, when investigating the stability of the motion of a disk with a non-restoring constraint, a method different from the one used earlier [5, 8] is proposed. The non-smooth change of variables of [5] is not used at all and the change of variables which is analogous to

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the "reducing" change of variables in [8] is used to reduce the boundary conditions in the case of collision to the simplest form, when the impulse corresponding to the coordinate on which the non-restoring coupling is imposed merely changes sign when the disk collides with the plane. The change of variables which reduces the Hamiltonian to action-angle variables in the case of non-perturbed motion is then made. The variable of action is not changed during collision and the analysis of the orbital stability reduces to an analysis of a Hamiltonian system with two degrees of freedom, periodic in the angular variable, in a time interval corresponding to free flight of the disk between two successive collisions with the plane.

1. THE EQUATIONS OF MOTION AND THEIR FIRST INTEGRALS

We will refer the motion of the disk to a fixed system of coordinates $OXYZ$ with the origin at an arbitrary point O of the plane above which the disk moves and with vertical OZ axis (Fig. 1). Let $Gxyz$ be the system of coordinates specified by the principal central axis of inertia of the disk, and let the axis Gz be perpendicular to the plane of the disk. We will specify the position of the disk by the coordinates x, y and z of its centre of gravity in the system of coordinates $OXYZ$ and by the three Euler angles ψ, θ and φ, specifying the orientation of the trihedron $Gxyz$ with respect to the König system of coordinates $GXYZ$.

The point M of the disk nearest to the OXY plane is above this plane throughout the motion or lies in it. Hence the inequality $z \geqslant R\sin\theta$ holds, where R is the radius of the disk. If we put $\zeta = z - R\sin\theta$ then the non-restoring constraint can be written in the form of the inequality $\zeta \geqslant 0$.

The Lagrange function

$$L = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \zeta^2) + mR\cos\theta\zeta\dot{\theta} + \tfrac{1}{8}mR^2(1 + 4\cos^2\theta)\dot{\theta}^2 + \\ + \tfrac{1}{8}mR^2\sin^2\theta\dot{\psi}^2 + \tfrac{1}{4}mR^2(\dot{\psi}\cos\theta + \dot{\varphi})^2 - mg(\zeta + R\sin\theta) \tag{1.1}$$

corresponds to free flight of the disk when $\zeta > 0$.

Here m is the mass of the disk and g is the acceleration due to gravity.

The condition for an absolutely elastic collision can be written in the form of the equality $\dot{\zeta}^+ = -\zeta^-$. Here and henceforth the values of corresponding quantities before and after collision are given the subscripts minus and plus.

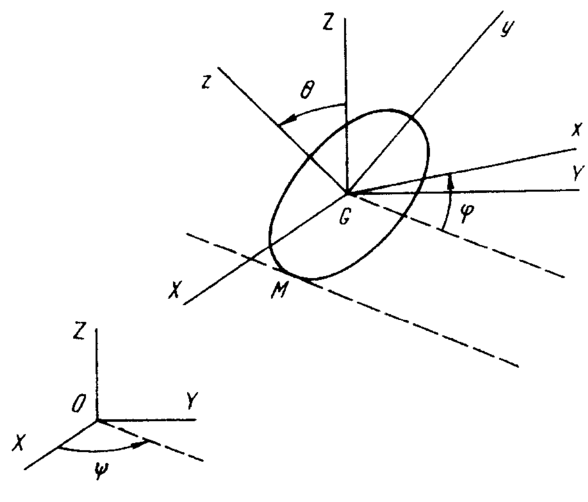

Fig. 1.

All the generalized coordinates x, y, ζ, ψ, θ, φ and all the generalized momenta, except the momentum p_ζ corresponding to the generalized coordinate ζ, remain constant on impact [14]. Noting also that the Lagrange function is independent of x, y, ψ, φ, we obtain the integrals

$$p_x = m\dot{x} = c_x = \text{const}, \quad p_y = m\dot{y} = c_y = \text{const} \tag{1.2}$$

$$p_\psi = \tfrac{1}{4}mR^2\sin^2\theta\dot{\psi} + \tfrac{1}{2}mR^2(\dot{\psi}\cos\theta + \dot{\varphi})\cos\theta = \tfrac{1}{4}mR^2\alpha(\alpha = \text{const}) \tag{1.3}$$

$$p_\varphi = \tfrac{1}{2}mR^2(\dot{\psi}\cos\theta + \dot{\varphi}) = \tfrac{1}{2}mR^2\beta \quad (\beta = \text{const})$$

which hold throughout the motion, including the intervals of free flight of the disk and the instants when it collides with the plane.

From (1.2) it follows that the projection of the disk centre of gravity onto the OXY plane moves uniformly and rectilinearly. Without loss of generality we shall assume that $c_x = c_y = 0$ in (1.2), i.e. the disk centre of gravity moves along the specified vertical. According to (1.3), the projections of the disk angular momentum about the centre of gravity onto the vertical and onto the axis of symmetry Gz are constant in the process.

In what follows it is convenient to use the Hamiltonian form of the equations of motion to describe the free flight of the disk. Setting $\theta = \pi/2 + q$ and introducing the momenta

$$p = \tfrac{1}{4}mR^2(1 + 4\sin^2 q)\dot{q} - mR\sin q\dot{\zeta} \tag{1.4}$$

$$p_\zeta = m\dot{\zeta} - mR\sin q\dot{q} \tag{1.5}$$

we obtain the Hamilton function

$$H = \frac{2}{mR^2}p^2 + \frac{4}{mR}\sin q p p_\zeta + \frac{1}{2m}(1 + 4\sin^2 q)p_\zeta^2 + \\ + \frac{mR^2}{8\cos^2 q}(\alpha + 2\beta\sin q)^2 + mg(\zeta + R\cos q) \tag{1.6}$$

It follows from the equality $\dot{\zeta}^+ = -\dot{\zeta}^-$ and the equation $\dot{\zeta} = \partial H/\partial p_\zeta$ that the values of the momentum p_ζ before and after collision are related by the formula

$$p_\zeta^+ = -p_\zeta^- - \frac{8\sin q}{R(1 + 4\sin^2 q)}p \tag{1.7}$$

The quantities q, p and ζ do not change during the collision

2. NON-PERTURBED MOTION. THE ACTION-ANGLE VARIABLES

It follows from the equations with Hamiltonian (1.6) and boundary conditions (1.7) that motions exist in which the disk plane is vertical and the disk itself rotates with a constant angular velocity about the horizontal or vertical axis.

In the first motion we have

$$q = 0, \quad \dot{\psi} = 0, \quad \dot{\varphi} = \omega_1(\theta = \pi/2, \quad \alpha = 0, \quad \beta = \omega_1) \tag{2.1}$$

In the second motion we have

$$q = 0, \quad \dot{\psi} = \omega_2, \quad \dot{\varphi} = 0 \ (\theta = \pi/2, \quad \alpha = \omega_2, \quad \beta = 0) \tag{2.2}$$

The motion of the disk centre of gravity then occurs along the fixed vertical and, when $\zeta > 0$, is described by the equations

$$d\zeta / dt = \partial\Gamma / \partial p_\zeta, \quad dp_\zeta / dt = -\partial\Gamma / \partial\zeta \tag{2.3}$$

$$\Gamma = \frac{1}{2m} p_\zeta^2 + mg\zeta \tag{2.4}$$

while the constraint of the values of the momentum p_ζ before and after a collision of the disk with the plane, obtained from (1.7) when $q = 0$, has the form

$$p_\zeta^+ = -p_\zeta^- \tag{2.5}$$

As the result of collisions between the disk and the plane its centre of gravity executes periodic motion with period $\tau = 2(2h/g)^{1/2}$ where h is the height of jump of the lowest point of the disk above the plane, and the quantity τ is equal to the time interval between two successive collisions. The trajectory shown in Fig. 2 corresponds to the periodic motion of the centre of gravity in the phase plane ζ, p_ζ. This trajectory is a parabola, given by the equation

$$p_\zeta^2 / (2m) + mg\zeta = mgh \tag{2.6}$$

If we assume that the collision occurs in the time "interval" from $t = -0$ to $t = +0$, we have $\zeta = 0$, $p_\zeta = p_\zeta^- = -m(2gh)^{1/2}$ when $t = -0$ and

$$\zeta(t) = -gt^2 / 2 + (2gh)^{1/2} t, \quad p_\zeta = m(2gh)^{1/2} - mgt \tag{2.7}$$

when $0 \leqslant t < \tau$.

On collision there is an abrupt change in momentum from p_ζ^- to p_ζ^+. The part of the phase trajectory lying on the axis $\zeta = 0$ corresponds to this variation in Fig. 2.

For what follows it is useful to describe the non-perturbed motion of the disk centre of gravity in terms of the variables I and W, where I is the action and W is the corresponding angular variable [15]. We have

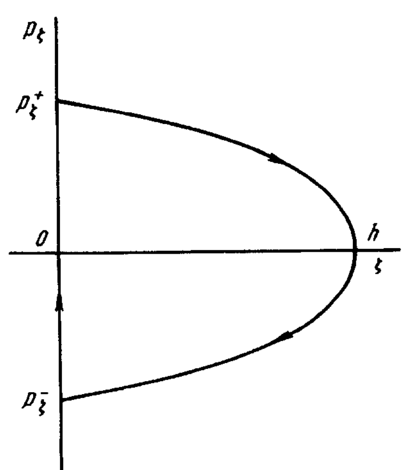

Fig. 2.

$$I = \frac{1}{2\pi}\oint p_\zeta d\zeta$$

where the line integral is computed along trajectory (2.6), whence we obtain an expression for the height of the disk jump in terms of the action variable in non-perturbed motion

$$h = \left(\frac{9\pi^2}{8m^2g}\right)^{1/3} I^{2/3} \tag{2.8}$$

The Hamilton function (2.4) in the action-angle variables is

$$\Gamma = \Gamma(I) = (9m\pi^2g^2/8)^{1/3} I^{2/3} \tag{2.9}$$

To obtain an explicit form of the canonical transformation p_ζ, $\zeta \to I$, W we will use the fact that the solution of Eqs (2.3) is known. This is specified by formulae (2.7). It is only necessary to replace h by I in them in accordance with formula (2.8) and to express the time in terms of the angular variable using the fact that $W = \omega(I)t$, and

$$\omega = \frac{2\pi}{\tau} = \pi\left(\frac{g}{2h}\right)^{1/2} = \frac{\partial\Gamma}{\partial I} = \left(\frac{m\pi^2g^2}{3I}\right)^{1/3}$$

The desired univalent canonical change of variables is 2π-periodic in W and is specified by the formulae

$$\zeta = \left(\frac{9}{8m^2\pi^4g}\right)^{1/3} I^{2/3}W(2\pi - W), \quad p_\zeta = \left(\frac{3m^2g}{\pi^2}\right)^{1/3} I^{1/3}(\pi - W) \tag{2.10}$$

when $0 \leqslant W < 2\pi$.

It is significant that the action variable is not changed on collision. This is obvious from (2.5) and from the formula for p_ζ in replacement (2.10). However, this follows directly from the geometrical meaning of this variable: I is the area divided by 2π bounded by the phase trajectory in Fig. 2, and this quantity is constant in the non-perturbed motion.

3. SIMPLIFICATION OF THE BOUNDARY CONDITIONS

We will investigate the orbital stability of the disk translation-rotational motions described in Section 2. This means that the stability of these motions with respect to the variables q, $\dot{q}$ and the height h of the disk jump above the plane will be considered. The constants α and β of integrals (1.3) are assumed to be non-perturbed. We will consider in detail the case when the disk rotates about the vertical in the non-perturbed motion.

We will first make the canonical change of variables ζ, q, p_ζ, $p \to Q$, ξ, P, η such that the coordinate ζ remains unchanged ($\zeta = Q$), the momentum η stays constant on collision and boundary conditions (1.7) takes a form analogous to relations (2.5) in the non-perturbed motion

$$P^+ = -P^- \tag{3.1}$$

We take the generating function of this replacement in the form

$$S = \zeta P + f(\zeta, q)\eta$$

where the function f is as yet unknown; it will be found from condition (3.1).

In implicit form the desired canonical transformation is given by the equalities

$$Q=\frac{\partial S}{\partial P}=\zeta,\quad \xi=\frac{\partial S}{\partial \eta}=f(\zeta,q),\quad p_\zeta=\frac{\partial S}{\partial \zeta}=P+\frac{\partial f}{\partial \zeta}\eta,\quad p=\frac{\partial S}{\partial q}=\frac{\partial f}{\partial q}\eta \tag{3.2}$$

Since $q^+=q^-$, $\zeta^+=\zeta^-$, $p^+=p^-$ the last relation of (3.2) gives the equality $\eta^+=\eta^-$. Taking (3.2) into account, condition (1.7) can be written in the form of the relation

$$p^+=-p^- -2\eta\left[\frac{\partial f}{\partial \zeta}+\frac{4\sin q}{R(1+4\sin^2 q)}\frac{\partial f}{\partial q}\right]$$

which, by the condition of the problem, must be equivalent to relation (3.1) for any η. Hence it follows that the function satisfying the linear homogeneous first-order partial differential equation

$$\frac{\partial f}{\partial \zeta}+\frac{4\sin q}{R(1+4\sin^2 q)}\frac{\partial f}{\partial q}=0$$

can be taken as the function $f(\zeta, q)$.

The general solution of this equation is an arbitrary differentiable function of the first integral of the corresponding ordinary differential equation [16].

Assuming that the replacement $\xi=f(\zeta,\ q)$ is identical (i.e. $f(0,\ q)=q$) when $\zeta=0$ we find that the function is given implicitly by the equality

$$\ln tg\frac{q}{2}-4\cos q-\frac{4}{R}\zeta=\ln tg\frac{\xi}{2}-4\cos\xi \tag{3.3}$$

For small values of q we have

$$f(\zeta,q)=\exp\left(-\frac{4\zeta}{R}\right)\left[q+\frac{25}{12}\left(1-\exp\left(-\frac{8\zeta}{R}\right)\right)q^3\right]+O(q^5)$$

and the change of variables $\zeta, q,\ p_\zeta,\ p\to Q,\ \xi, P, \eta$ is given by the series

$$\begin{aligned}
&f(\zeta,q)=\exp\left(-\frac{4\zeta}{R}\right)\left[q+\frac{25}{12}\left(1-\exp\left(\frac{8\zeta}{R}\right)\right)q^3\right]+O(q^5)\\
&p=\exp\left(-\frac{4Q}{R}\right)\left[\eta+\frac{25}{4}\left(1-\exp\left(-\frac{8Q}{R}\right)\right)\xi^2\eta\right]+\ \dots\\
&\zeta=Q,\quad p_\zeta=P-\frac{4}{R}\xi\eta+\frac{50}{3R}\xi^3\eta\ +\ \dots
\end{aligned} \tag{3.4}$$

The dots here denote terms whose power with respect to ξ and η is not less than five.

4. THE HAMILTON FUNCTION OF PERTURBED MOTION

In new variables Hamilton function (1.6) (when $\alpha=\omega_2$, $\beta=0$) can be represented by a series in powers of ξ and η, whose coefficients are functions of Q and P. This series does not contain terms which are linear in ξ and η and, when $\xi=\eta=0$, is the same as the right-hand side of expression (2.4) in which $p_\zeta=P$, $\zeta=Q$.

Now let us introduce the canonical conjugate variables J and υ instead of the variables P and Q by the formulae which are analogous to (2.10)

$$P=(3m^2\pi g)^{1/3}J^{1/3}f_1(\upsilon),\quad Q=\left(\frac{9\pi^2}{8m^2g}\right)^{1/3}J^{1/3}f_2(\upsilon), \tag{4.1}$$

where f_1 and f_2 are the 2π-periodic functions. We have

$$f_1=1-\upsilon/\pi,\quad f_2=\upsilon/\pi(2-\upsilon/\pi) \tag{4.2}$$

when $0\leqslant\upsilon<2\pi$.

In non-perturbed motion J and υ are the action-angle variables I and W introduced in Section 2.

To investigate the stability we will take as the perturbations the dimensionless quantities x_1, x_2 and r, introduced by means of the canonical transformation (with valency $4/(mR^2\omega_2)$)

$$\xi=x_1,\quad \eta=\tfrac{1}{4}mR^2\omega_2x_2,\quad \upsilon=\upsilon,\quad J=I+\tfrac{1}{4}mR^2\omega_2r \tag{4.3}$$

The stability with respect to the variables x_1, x_2 and r means the orbital stability of the periodic motion of the disk. It is essential that all three quantities x_1, x_2 and r are not changed during the collision.

In addition, if we change to dimensionless time ω_2t we obtain, after quite lengthy calculations using (1.6), (3.4) and (4.1)–(4.3), the Hamilton function of the perturbed motion in the form of a series given in powers of the quantities x_1, x_2 and r with coefficients 2π-periodic in υ

$$H=H_2+H_4+\ldots \tag{4.4}$$

where H_k is the form of power k relative to x_1, x_2, $|r|^{1/2}$. In expansion (4.4), there are no forms of odd powers

$$H_2=\frac{\pi}{a}r+\frac{1}{2f_3}(f_3^2f_4x_1^2+x_2^2) \tag{4.5}$$

$$H_4=c_{20}r^2+(h_{20}x_1^2+h_{02}x_2^2)r+h_{22}x_1^2x_2^2+h_{40}x_1^4 \tag{4.6}$$

Here

$$\begin{gathered}f_3=\exp(a^2bf_2),\quad f_4=a^2b^2f_1^2-b+1,\quad c_{20}=-2\pi^2/(a^4b^2)\\ h_{20}=4\pi f_3(bf_1^2+f_2f_4)/(ab),\quad h_{02}=-4\pi f_2/(abf_3)\\ h_{22}=\tfrac{1}{4}(17-25/f_3),\quad h_{40}=f_3[2(25-21f_3)+2a^2b^2f_1^2(25-27f_3)-b(50-51f_3)]/24\end{gathered} \tag{4.7}$$

The dimensionless parameters a and b, on which Hamiltonian (4.4) depends, are defined by the equalities

$$a=\omega_2(2h/g)^{1/2},\quad b=4g/(\omega_2^2R) \tag{4.8}$$

5. THE STABILITY IN THE FIRST APPROXIMATION. CHARACTERISTIC EXPONENTS

The first approximation is described by the equations with Hamiltonian (4.5). If one takes the angular variable υ as the independent variable in these equations then the equations for x_1 and x_2 are separated. The Hamiltonian 2π-periodic in υ of the form

$$h_2 = \frac{a}{2\pi f_3}(f_3^2 f_4 x_1^2 + x_2^2) \tag{5.1}$$

corresponds to them.

To investigate the stability in the first approximation it is necessary to compute the fundamental matrix $X(\upsilon)$ of solutions of these equations. The explicit form of this matrix cannot be found in the general case of a linear system with periodic coefficients. In the problem under consideration it can be found if one uses the fact that free flight of the disk above the plane corresponds to the variation of the variable υ from 0 to 2π. During this flight, according to equalities (1.4) and (1.5) and the equations of motion with Hamiltonian (1.6), the relations

$$\ddot{q} + \omega_2^2 q = 0, \quad p = \tfrac{1}{4} mR^2 \dot{q} - m(2gh)^{1/2} Rf_1(\upsilon)q$$

are satisfied in the linear approximation in q and p.

In addition, taking the equality $\upsilon = \pi\omega_2 t/a$ and formulae (3.4) and (4.3) of the replacements of variables into account, we find the general solution of the linear equations with Hamiltonian (5.1) in the form

$$x_1 = f_3^{-1/2}\left(c_1 \sin\frac{a\upsilon}{\pi} + c_2 \cos\frac{a\upsilon}{\pi}\right)$$

$$x_2 = f_3^{1/2}\left[c_1\left(\cos\frac{a\upsilon}{\pi} - abf_1 \sin\frac{a\upsilon}{\pi}\right) - c_2\left(\sin\frac{a\upsilon}{\pi} + abf_1 \cos\frac{a\upsilon}{\pi}\right)\right]$$

where c_1 and c_2 are arbitrary constants.

The elements x_{ij} of the matrix $\mathbf{X}(\upsilon)$ satisfying the condition $\mathbf{X}(0) = \mathbf{E}$, where $\mathbf{E}$ is the identity matrix, will be

$$x_{11} = f_3^{-1/2}\left(\cos\frac{a\upsilon}{\pi} + ab\sin\frac{a\upsilon}{\pi}\right), \quad x_{12} = f_3^{-1/2}\sin\frac{a\upsilon}{\pi}$$
$$x_{21} = f_3^{1/2}\left[ab(1-f_1)\cos\frac{a\upsilon}{\pi} - (1 + a^2b^2 f_1)\sin\frac{a\upsilon}{\pi}\right], \quad x_{22} = f_3^{1/2}\left(\cos\frac{a\upsilon}{\pi} - abf_1 \sin\frac{a\upsilon}{\pi}\right) \tag{5.2}$$

The characteristic equation of the matrix $\mathbf{X}(2\pi)$ can be represented in the form

$$\rho^2 - 2A\rho + 1 = 0 \qquad A = \cos 2a + ab\sin 2a \tag{5.3}$$

If $A > 1$ then Eq. (5.3) has a root whose modulus is greater than unity. In this case the periodic motion of the disk in question is unstable (regardless of the linear terms in the equations of perturbed motion). If $|A| < 1$ then both roots have moduli equal to unity and the stability (in the first approximation) occurs [17].

After some reduction, we obtain that the condition $|A| < 1$ is equivalent to the set of two systems of inequalities

$$-\mathrm{ctg}\,a < ab < \mathrm{tg}\,a, \quad \mathrm{tg}\,a < ab < -\mathrm{ctg}\,a \tag{5.4}$$

In the plane of the parameters a and b there is a denumerable set of domains of instability and of the stability in the first approximation. They are shown in Fig. 3. The domains of instability are shown hatched. The domains of the stability are denoted by the digits, 1, 2, 3, ... The domain with number l is bounded below by the segment of the axis $b = 0$ on which $\pi(l-1)/2 < a < \pi l/2$, the straight line $a = \pi l/2$ is its right boundary while the curve $b = \mathrm{tg}\,a/a$ is

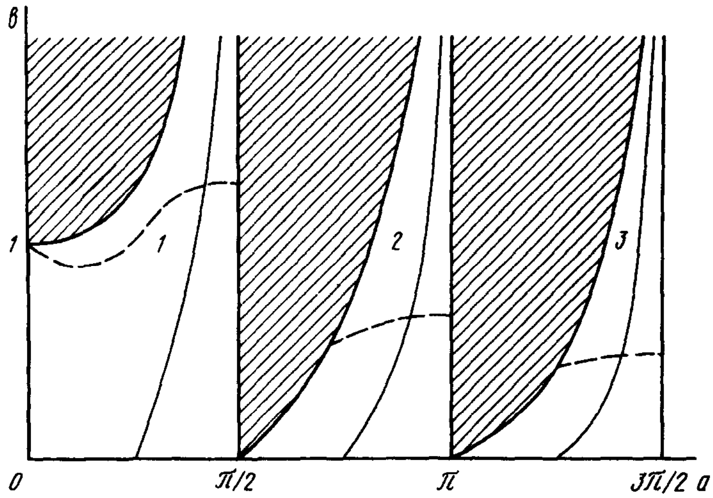

Fig. 3.

its curvilinear boundary when l is odd and the curve $b=-\mathrm{ctg}\,a/a$ plays the same role when l is even but in the domain themselves $b<\mathrm{tg}\,a/a$ and $b<-\mathrm{ctg}\,a/a$ respectively.

In the domains of the stability in the first approximation the roots of Eq. (5.3) can be written in the form $\rho_{1,2}=\exp(\pm i\lambda)$ where $\pm i\lambda$ are the characteristic exponents. From (5.3) it follows that $\cos 2\pi\lambda = A$. From equality λ is not defined uniquely. The ambiguity is removed if one uses the fact that λ is continuous with respect to the parameters. In fact, if one puts $b=0$ in Hamiltonian (5.1) we obtain the Hamiltonian of a harmonic oscillator with frequency $\lambda = a/\pi$. Therefore, in the domain with number l we have

$$\lambda = \begin{cases} \dfrac{1}{2\pi}\arccos A + \dfrac{1}{2}(l-1), & l \text{ is odd} \\ -\dfrac{1}{2\pi}\arccos A + \dfrac{1}{2}l, & l \text{ is even} \end{cases} \tag{5.5}$$

6. THE NON-LINEAR PROBLEM OF STABILITY

We will now consider the non-linear problem of the orbital stability of the periodic motion of the disk for values of the parameters a and b lying in the domains of its stability in the first approximation. To do this it is first necessary to reduce the Hamilton function (5.1) to normal form

$$h_2 = \frac{1}{2}\lambda(y_1^2 + y_2^2) \tag{6.1}$$

by means of the 2π-periodic in υ linear canonical transformation $x_1,\ x_2 \to y_1,\ y_2$, which corresponds to harmonic oscillations with frequency λ computed by formulae (5.5). Since the fundamental matrix $X(\upsilon)$ of solutions has been found in explicit form then, according to the algorithm of [18], the transformation $x_1,\ x_2 \to y_1,\ y_2$ can also be obtained in explicit form

$$x_1 = n_{11}(\upsilon)y_1 + n_{12}(\upsilon)y_2, \quad x_2 = n_{21}(\upsilon)y_1 + n_{22}(\upsilon)y_2, \tag{6.2}$$

$$\begin{aligned} n_{11} &= \kappa x_{12}\cos\lambda\upsilon - \kappa^{-1}x_{11}\sin\lambda\upsilon, \quad n_{12} = -\kappa x_{12}\sin\lambda\upsilon - \kappa^{-1}x_{11}\cos\lambda\upsilon, \\ n_{21} &= \kappa x_{22}\cos\lambda\upsilon - \kappa^{-1}x_{21}\sin\lambda\upsilon, \quad n_{22} = -\kappa x_{22}\sin\lambda\upsilon - \kappa^{-1}x_{21}\cos\lambda\upsilon, \end{aligned} \tag{6.3}$$

$$\kappa = (\sin 2\pi\lambda / \sin 2a)^{1/2}$$

The functions $x_{ij}(\upsilon)$ are defined by equalities (5.2).

The canonical transformation (6.2) can be specified using the generating function S_1. By the theory of canonical transformation, the Hamilton function (5.1), its normal form and the function S_1 are related by the identity

$$\frac{1}{2}\lambda(y_1^2+y_2^2) = \frac{a}{2\pi f_3}\left[f_3^2 f_4(n_{11}y_1+n_{12}y_2)^2+(n_{21}y_1+n_{22}y_2)^2\right]+\frac{\partial S_1}{\partial \upsilon} \tag{6.4}$$

If we now take the generating function $S_2 = r_1\upsilon + S_1$ then equalities (6.2), supplemented by the formulae

$$\upsilon = \frac{\partial S_2}{\partial r_1} = \upsilon, \quad r = \frac{\partial S_2}{\partial \upsilon} = r_1 + \frac{\partial S_1}{\partial \upsilon} \tag{6.5}$$

(the derivative $\partial S_1/\partial\upsilon$ is specified by identity (6.4)) will give the canonical univalent transformation $\upsilon, r, x_1, x_2 \to \upsilon_1, r_1, y_1, y_2$ of all phase variables on which the Hamilton function (4.4) of the perturbed motion depends.

Substituting x_1, x_2 and r obtained from (6.2)–(6.5) as functions of the variables y_1, y_2, r_1, υ into (4.4) we find the Hamilton function of the perturbed motion in the form

$$H = \frac{\pi}{a}r_1 + \frac{\pi\lambda}{2a}(y_1^2+y_2^2) + c_{20}r_1^2 + K_2 r_1 + K_4 + \ldots \tag{6.6}$$

where K_m is the form of the power m relative to y_1, y_2 with the 2π-periodic in υ coefficients

$$K_m = \sum_{i+j=m} k_{ij}y_1^i y_2^j, \quad K_2 = 2c_{20}\frac{\partial S_1}{\partial \upsilon} + h_{20}x_1^2 + h_{02}x_2^2$$

$$K_4 = c_{20}\left(\frac{\partial S_1}{\partial \upsilon}\right)^2 + (h_{20}x_1^2 + h_{02}x_2^2)\frac{\partial S_1}{\partial \upsilon} + h_{22}x_1^2x_2^2 + h_{40}x_1^4$$

where $\partial S_1/\partial\upsilon$, x_1 and x_2 are defined by (6.2)–(6.4). The dots in (6.6) denote the terms of the series whose power relative to y_1, y_2 and r_1 is greater than five.

We must now normalize the terms in (6.6) of the fourth power. The normal form will be different depending on whether resonance of the fourth power ($4\lambda = n$ is an integer) exists in the system or not. It follows from (5.3) and (5.5) that this resonance occurs when the parameters a and b take values satisfying the equality $ab = -\operatorname{ctg} 2a$. The corresponding resonance curve exists in each of the denumerable set of domains of stability in the first approximation. In Fig. 3 the resonance curves are represented by the thin solid lines.

A near-identical, 2π-periodic in υ canonical transformation $r_1, \upsilon, y_1, y_2 \to \mu_1, \nu_1, z_1, z_2$ normalizing the terms of the fourth power in (6.6) can be found by one of the Deprit–Hori type methods [18] or using classical perturbation theory. Computations show that, when there is no resonance $4\lambda = n$, Hamiltonian (6.6) normalized up the terms of the fourth power inclusive has the form

$$H = \frac{\pi}{a^{-1}}(\mu_1 + \lambda\mu_2) + c_{20}\mu_1^2 + c_{11}\mu_1\mu_2 + c_{02}\mu_2^2 + O\left((|\mu_1| + \mu_2)^3\right) \tag{6.7}$$

$$(z_1 = (2\mu_2)^{1/2}\sin\nu_2, \quad z_2 = (2\mu_2)^{1/2}\cos\nu_2)$$

$$c_{11} = \frac{1}{2\pi}\int_0^{2\pi}(k_{20}+k_{02})d\upsilon, \quad c_{02} = \frac{1}{4\pi}\int_0^{2\pi}(3k_{40}+k_{22}+3k_{04})d\upsilon$$

The quantity c_{20} is defined in (4.7).

When the inequality

$$\delta \equiv c_{20}\lambda^2 - c_{11}\lambda + c_{02} \neq 0 \tag{6.8}$$

holds, the motion in which $\mu_1 = \mu_2 = 0$ is stable in Lyapunov's sense with respect to the variables μ_1 and μ_2 [19, 20]. Hence the orbital stability of the periodic motion of the disk follows in this problem.

But if the parameters a and b lie on the resonance curve $4\lambda = n$ then in (6.7) terms of the form

$$\mu_2^2[d\sin(nv_1 - 4v_2) + e\cos(nv_1 - 4v_2)]$$

where

$$d = \frac{1}{4\pi}\int_0^{2\pi}[(k_{40} - k_{22} + k_{04})\sin n\upsilon - (k_{13} - k_{31})\cos n\upsilon]d\upsilon$$

$$e = \frac{1}{4\pi}\int_0^{2\pi}[(k_{40} - k_{22} + k_{04})\cos n\upsilon + (k_{13} - k_{31})\sin n\upsilon]d\upsilon$$

are added to the terms of the second power in μ_1 and μ_2.

When the inequality

$$|\delta| < (d^2 + e^2)^{1/2} \tag{6.9}$$

is satisfied instability occurs. The periodic motion of the disk is orbitally stable if the opposite inequality to (6.9) holds.

An asymptotic analysis, carried out for small values of the parameter b, showed that the estimates

$$\delta = -2(ab)^{-2} + O(b^{-1}), \quad (d^2 + e^2)^{1/2} = O(1)$$

hold as $b \to 0$, whence it follows that for sufficiently small b the periodic motion of the disk in question is orbitally stable whether or not resonance exists.

For arbitrary values of the parameters a and b inequalities (6.8) and (6.9) were checked on a computer for the first three $(i = 1, 2, 3)$ regions of stability in the first approximation. It was found that there are instability sections on the resonance curves. These sections in the first, second and third regions are given by the inequalities $1.18188 < b < 1.26758$, $0.61620 < b < 0.63659$ and $0.41574 < b < 0.42454$, respectively. For the problem of stability to be solved at the boundary points of these intervals one must take into account terms of powers higher than the fourth in the series expansion of the Hamiltonian of the perturbed motion. For non-resonance values of the parameters a and b, lying on the resonance curves, where inequality (6.8) is not satisfied, we have an analogous situation. These curves are shown in Fig. 3 by the dashed line. They intersect the resonance curves in the instability sections mentioned above. The periodic motion of the disk is orbitally stable for the remaining values of the parameters a and b of the first three regions of the stability in the first approximation.

Note. It follows from (1.6) that if the unimportant constant is neglected the Hamiltonian of the perturbed motion for motion (2.2)–(2.5) considered becomes the Hamiltonian of the perturbed motion for (2.1), (2.3)–(2.5) when $2\omega_1$ is substituted for ω_2. All the conclusions on stability, obtained for the periodic motion of the disk when it rotates about the vertical, can be extended, by a simple change in the parameters a and b specified by equalities (4.8), to the case in which the rotation occurs about the horizontal axis.

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